J. DIFFERENTIAL GEOMETRY
7 (1971) 229-256

NOVIKOV'S HIGHER SIGNATURE AND FAMILIES OF ELLIPTIC OPERATORS

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Introduction

Let X be a $2k$-dimensional connected manifold[1], and suppose that $x_1, \cdots, x_n$ is a basis for $H^1(X; Z)$; let $\xi_1, \cdots, \xi_n$ be the dual basis for Hom $(H^1(X; Z), Z)$. For any set of indices $\{1 \leq i_1 < \cdots < i_r \leq n\}$ with $r \equiv 2k \pmod 4$ consider a submanifold $M_{i_1 \cdots i_r} \subset X$ with trivial normal bundle dual to the product $x_{i_1} \cdots x_{i_r}$. Define the Novikov higher signature of X by the expression

$$(*) \qquad \sum_{r \equiv 2k(4)} \sum_{0 \leq i_1 < \cdots < i_r \leq n} 2^{r/2} \operatorname{sign}(M_{i_1 \cdots i_r}) \xi_{i_1} \wedge \cdots \wedge \xi_{i_r} \in \Lambda \operatorname{Hom}(H^1(X; Z), Z) .$$

Here sign $(M_{i_1 \cdots i_r})$ denotes the usual Hirzebruch signature. (Note that our definition differs slightly from the one of W. C. Hsiang [6] by the presence of the factors $2^{r/2}$.)

It is easy to see that (*) is independent of the choice of basis. Novikov conjectured that the expression (*) is a homotopy invariant of the manifold X and provided evidence [11] in favor of this conjecture. Rohlin [14] has obtained further partial results. The proof in general has been obtained by W. C. Hsiang-Farrell [6] and Kasparov [7] using nonsimply connected surgery.

One of the results of this paper is a new proof of Novikov's conjecture based on a completely different approach.

For any Kähler manifold X there is an associated complex torus Pic (X) whose points are the isomorphism classes of holomorphic line bundles on X which are topologically trivial. Let L_p denote the holomorphic line bundle corresponding to $p \in$ Pic (X). Then there is a holomorphic line bundle L over $X \times$ Pic (X) such that for a given $p \in$ Pic (X), the restriction $L | X \times \{p\}$ is isomorphic to L_p. (See [8].)

Let $\Lambda^p T^* X$ be the p-th exterior power of the holomorphic cotangent bundle of X, and let $\mathcal{O}(\Lambda^p T^* X \otimes L)$ be the sheaf of holomorphic sections of $\Lambda^p T^* X \otimes L$

Received July 23, 1971. Supported in part by NSF Grant GP-7952X1. This is essentially the author's Ph.D. Thesis at Princeton University (May, 1971).

[1] Throughout this paper the word "manifold" means "closed oriented smooth manifold".

over $X \times \text{Pic}(X)$. In analogy with the Hodge signature theorem we form the element

$$(**) \qquad \sum_{p,q} (-1)^q R^q \pi_*(\mathcal{O}(\Lambda^p T^* X \otimes L))$$

belonging to the K-theory of coherent analytic sheaves on $\text{Pic}(X)$. (Here $\pi: X \times \text{Pic}(X) \to \text{Pic}(X)$ is the projection, and $R^q\pi_*$ denote the higher direct image functors.) A calculation based on the Riemann-Roch theorem for the map π shows that the Chern character of (**) is equal to (*), provided we identify $\Lambda\,\text{Hom}(X; \mathbf{Z}), \mathbf{Z})$ with $H^*(\text{Pic}(X); \mathbf{Z})$.

Actually, we show that (*) has an analytic interpretation even if X has not a complex structure. For arbitrary X define a real torus $\text{Pic}(X)$ as the identity component of the group of homomorphisms of $\pi_1(X; x_0)$ into the unit circle S^1. (x_0 is a base point of X.) One sees easily that for X Kähler, there is a natural isomorphism from this real torus to the one constructed from the complex structure. Every point $p \in \text{Pic}(X)$ defines an induced complex flat line bundle L_p on X with a flat positive definite hermitian metric, and there is a complex line bundle L over $X \times \text{Pic}(X)$ such that for any $p \in \text{Pic}(X)$, $L \mid X \times \{p\}$ is given a flat structure and a flat hermitian metric which vary continuously with p, and $L \mid X \times \{p\} \approx L_p$ the isomorphism being compatible with the flat structures and hermitian metrics (see § 4.1, § 4.2). Consider a riemannian metric on X. Associated to it there is an elliptic operator D on X whose index equals sign (X), see [2, § 6]. For any $p \in \text{Pic}(X)$, we can "twist" D by tensoring it with the flat hermitian bundle $L \mid X\{p\}$ and obtain a new elliptic operator D_p. The collection $\{D_p\}_{p \in \text{Pic}(X)}$ is then a family of elliptic operators [3] parametrized by $\text{Pic}(X)$ whose analytical index $\text{Nov}(X) \in K(\text{Pic}(X))$ has the property that its Chern character equals (*). This is shown by applying the Atiyah-Singer index theorem for families of elliptic operators [3] which in our case is a substitute for the Riemann-Roch theorem when no complex structures are available.

We have hence a K-theory equivalent, $\text{Nov}(X)$ of (*) which turns out to be much easier to handle than (*). The advantages of the K-theory or analytic interpretation of Novikov's higher signature are:

(i) it leads to a direct proof of its homotopy invariance,

(ii) it can be defined apriori for any Poincaré complex,

(iii) it can be generalized for fibre-bundles with fibre X.

The paper is divided in 5 sections. In § 1 we study "hermitian complexes" over a compact parameter space $K(Y)$. These are complexes of vector bundles which satisfy Poincaré duality in a homotopy sense. A signature invariant in $K(Y)$ is introduced, which is shown to be a homotopy invariant of the hermitian complex (see Prop. 1.6). The hermitian complexes occur when dealing with families of signature operators; they are a more refined object than the analytic index of such families and play an essential role in our proofs. In § 2 we consider a generalization of Hirzebruch's signature theorem for the case when the coefficients are taken in a flat hermitian bundle. This cannot be proved by cobordism methods as in the classical case, but the general index theorem has to be used. An interesting consequence is derived in § 5 where we prove that if Γ is a discrete, torsion free subgroup of the real symplectic group Sp $(2n, \boldsymbol{R})$ with compact quotient, then for a manifold which is a $K(\Gamma, 1)$ the $\tilde{\mathscr{L}}$-classes are homotopy invariants in degrees $4k > n(n+1) - (n+2)/4$. This example is completely different from the case of a torus, when the $\tilde{\mathscr{L}}$-classes are also known to be homotopy invariants. Actually Novikov conjectured in [12] that the Pontrjagin classes of any manifold which is a $K(\pi, 1)$ are homotopy invariants. Our result shows that this is the case in certain range for $\pi = \Gamma$. § 3 is devoted to the proof of fibre-homotopy invariance of the analytic index of a family of signature operators with coefficients in a variable flat hermitian bundle (Theorem 3.3). This proof is carried out in more generality than needed for the particular case of the Novikov signature (§ 4.2) so that it can be extended to fibre bundles (§ 4.5, § 4.6). In § 4.3 we show that the invariant Nov $(X) \in K(\text{Pic}\,(X))$ admits an a priori definition for any Poincaré complex, whose Chern character, in case X is smooth, is equal to (*). This answers a question of Novikov [12], [13]. Note that Misčenko [10] has independently constructed an a priori invariant $M(X)$ belonging to a surgery obstruction group $L_{2k}(\boldsymbol{Z}^n)$ with coefficients in $\boldsymbol{Z}[\frac{1}{2}]$ but did not show its relation to Novikov's higher signature. The actual relation is as follows: from the group $L_{2k}(\boldsymbol{Z}^n)$ one should pass to K (character group of $\boldsymbol{Z}^n$) using the homomorphism of Gelfand-Misčenko [5]; this element in K-theory has to be interpreted as the analytic index of a family of elliptic operators, and then its Chern character can be calculated from the index theorem for families and gives (*).

In § 4.6 and § 4.7 we prove a multiplicativity formula for Novikov's higher signature: If $X \to Z \to Y$ is a smooth fibre bundle with X, Y, Z manifolds of even dimension such that Y is 2-connected, then

$$\text{Nov}\,(Z) = \text{Nov}\,(X)\cdot \text{sign}\,(Y) \in K(\text{Pic}\,(X)) = K(\text{Pic}\,(Z)).$$

It seems that this cannot be deduced by the method of [4].

I want to thank William Browder for accepting to be my thesis advisor and for his friendly interest in this work. I am indebted to Johan Dupont for several useful conversations and for his reading of the manuscript. Finally, I want to express my deep gratitude to Michael Atiyah for his constant support and encouragement. I have had the privilege of numerous conversations with him, which have strongly influenced this work.

1. Hermitian complexes

1.1. In this section Y will denote some compact space; all vector bundles over Y are supposed to be complex, and unless otherwise specified, finite dimensional. If V is a vector bundle over Y, then V' denotes the antidual bundle. If $f: V \to W$ is a morphism of vector bundles over Y, then $f': W' \to V'$ denotes the antidual morphism.

1.2. Hermitian complexes. A *hermitian complex* (over Y) is a bounded complex of vector bundles over Y:

$$\mathscr{C}: \{\cdots \longrightarrow \mathscr{C}^{-1} \xrightarrow{d} \mathscr{C}^0 \xrightarrow{d} \mathscr{C}^1 \longrightarrow \cdots\}$$

together with a homotopy-equivalence of complexes $\Phi: \mathscr{C} \to \mathscr{C}'$. (Here $\mathscr{C}'$ denotes the antidual complex defined by $(\mathscr{C}')^i = (\mathscr{C}^{-i})'$ with the antidual differentials. There will be no signs involved.) All morphisms of complexes are supposed to have degree zero. In particular, $\Phi: \mathscr{C}^i \to (\mathscr{C}^{-i})'$. We assume that Φ is symmetric, i.e., $\Phi' = \Phi$. The hermitian complex $\mathscr{C}$ will sometimes be denoted by $(\mathscr{C}, \Phi)$. Φ can also be considered as a pairing $\mathscr{C}^i \times \mathscr{C}^{-i} \xrightarrow{\langle , \rangle} 1$, (1 is the trivial bundle over Y) which is linear in the first variable and antilinear in the second, and satisfies $\langle x_1, x_2 \rangle = \overline{\langle x_2, x_1 \rangle}$. For this reason, Φ is called the *cup-product*. The condition that $\Phi: \mathscr{C} \to \mathscr{C}'$ is a homotopy equivalence is equivalent to the following condition: for any $y \in Y$ and any i the pairing of the i-th and $(-i)$-th cohomology space of the complex $\{\cdots \to \mathscr{C}_y^{-1} \to \mathscr{C}_y^0 \to \mathscr{C}_y^1 \to \cdots\}$ into $\boldsymbol{C}$, induced by Φ, is nonsingular.

This follows from the following

Lemma. *Let $\mathscr{C}$, $\mathscr{D}$ be two bounded complexes of vector bundles over Y, and $f: \mathscr{C} \to \mathscr{D}$ a morphism. Then f is a homotopy equivalence if and only if, for any $y \in Y$, the restriction $f_y: \mathscr{C}_y \to \mathscr{D}_y$ induces isomorphism in the cohomology.*

Proof. Using a mapping cylinder we can reduce to the case $\mathscr{C} = 0$. Hence we have to prove that if $\mathscr{D}_y$ is acyclic for any $y \in Y$, then $\mathscr{D}_y$ is homotopically trivial. Let d be the differential of $\mathscr{D}$. Choose positive definite hermitian metrics on the bundles of $\mathscr{D}$, and let d^* denote the adjoint of d with respect to these metrics. From the assumption it follows that $dd^* + d^*d$ is invertible on any fibre. It follows that $dd^* + d^*d$ is globally invertible. The operator $k = d^*(dd^* + d^*d)^{-1}$ satisfies then $kd + dk = 1$, and hence the lemma is proved.

Two hermitian complexes $(\mathscr{C}, \Phi)$ and $(\mathscr{D}, \Psi)$ over Y are said to be homotopy equivalent if there is a homotopy equivalence of ordinary complexes $g: \mathscr{C} \to \mathscr{D}$ such that Φ is homotopic to $g'\Psi g$. A hermitian complex is said to be *regular* if Φ is an isomorphism.

1.3. Proposition. *For any hermitian complex $(\mathscr{C}, \Phi)$ over Y there are a regular hermitian complex $(\hat{\mathscr{C}}, \hat{\Phi})$ and a homotopy equivalence $h: \mathscr{C} \to \hat{\mathscr{C}}$ admitting a left inverse and such that $\Phi = h'\hat{\Phi}h$.*

Proof. First suppose that $\Phi: \mathscr{C}^i \to (\mathscr{C}^{-i})'$ is an isomorphism for all $|i| \geq 2$. Consider the diagram

$$
\begin{array}{ccccc}
\mathscr{C}^0 & \xrightarrow{\ d\ } & \mathscr{C}^1 & \xrightarrow{\ d\ } & \mathscr{C}^2 \\
\Phi\big\downarrow & & \Phi\big\downarrow\ \big\uparrow\alpha & & \Phi\big\downarrow \\
(\mathscr{C}^0)' & \underset{\beta}{\overset{d'}{\rightleftarrows}} & (\mathscr{C}^{-1})' & \xrightarrow{\ d'\ } & (\mathscr{C}^{-2})'
\end{array}
$$

Using the fact that $\Phi\colon \mathscr{C}^2 \to (\mathscr{C}^{-2})'$ is an isomorphism one can easily find morphisms $\alpha\colon (\mathscr{C}^{-1})' \to \mathscr{C}^1$, $\beta\colon (\mathscr{C}^{-1})' \to (\mathscr{C}^0)'$ such that $\Phi\alpha + d'\beta = 1_{(\mathscr{C}^{-1})'}$.

Consider the diagram:

$$
(*)\qquad
\begin{array}{ccc}
\big\downarrow d & & \big\downarrow d' \\
\mathscr{C}^{-2} & \xrightarrow{\ \Phi\ } & (\mathscr{C}^2)' \\
\big\downarrow d\oplus 0 & & \big\downarrow d'\oplus 0 \\
\mathscr{C}^{-1}\oplus(\mathscr{C}^1)' & \xrightarrow{\ \tilde{\Phi}_{-1}\ } & (\mathscr{C}^1)\oplus\mathscr{C}^{-1} \\
\big\downarrow d\oplus 0\oplus 1 & & \big\downarrow d'\oplus 1\oplus 0 \\
\mathscr{C}^0\oplus(\mathscr{C}^{-1})'\oplus(\mathscr{C}^1)' & \xrightarrow{\ \tilde{\Phi}_0\ } & (\mathscr{C}^0)'\oplus\mathscr{C}^{-1}\oplus\mathscr{C}^1 \\
\big\downarrow d\oplus 1\oplus 0 & & \big\downarrow d'\oplus 0\oplus 1 \\
\mathscr{C}^1\oplus(\mathscr{C}^{-1})' & \xrightarrow{\ \tilde{\Phi}_1\ } & (\mathscr{C}^{-1})'\oplus\mathscr{C}^1 \\
\big\downarrow d\oplus 0 & & \big\downarrow d'\oplus 0 \\
\mathscr{C}^2 & \xrightarrow{\ \Phi\ } & (\mathscr{C}^{-2})' \\
\big\downarrow d & & \big\downarrow d' \\
\vdots & & \vdots
\end{array}
$$

where $\tilde{\Phi}_{-1}$, $\tilde{\Phi}_0$, $\tilde{\Phi}_1$ are defined by the matrices:

$$
\tilde{\Phi}_{-1} = \begin{pmatrix} \Phi & 1 \\ (d'\beta)' & -\alpha' \end{pmatrix}, \qquad
\tilde{\Phi}_0 = \begin{pmatrix} \Phi & \beta & d' \\ \beta' & 0 & -\alpha' \\ d & -\alpha & 0 \end{pmatrix}, \qquad
\tilde{\Phi}_1 = \begin{pmatrix} \Phi & d'\beta \\ 1 & -\alpha \end{pmatrix}.
$$

The first vertical line in (*) is a complex $\tilde{\mathscr{C}}$ such that the canonical imbedding $j\colon \mathscr{C} \to \tilde{\mathscr{C}}$ and the canonical projection $p\colon \tilde{\mathscr{C}} \to \mathscr{C}$ are homotopy equivalences and $p\cdot j = 1$. The horizontal maps define a morphism $\tilde{\Phi}\colon \tilde{\mathscr{C}} \to (\tilde{\mathscr{C}})'$ clearly satisfying $(\tilde{\Phi})' = \tilde{\Phi}$ and $\Phi = j'\cdot\tilde{\Phi}\cdot j$. It follows that $\tilde{\Phi}$ is a homotopy equivalence. Hence $(\tilde{\mathscr{C}}, \tilde{\Phi})$ is a hermitian complex. Moreover, $\tilde{\Phi}\colon \tilde{\mathscr{C}}^1 \to (\tilde{\mathscr{C}}^{-1})'$ is an isomorphism with inverse given by the matrix:

$$
\begin{pmatrix} \alpha & 1-\alpha\Phi \\ 1 & -\Phi \end{pmatrix}.
$$

Hence $\tilde{\Phi}: \tilde{\mathscr{C}}^{-1} \to (\tilde{\mathscr{C}}^{1})'$ is also an isomorphism. Suppose now that we only know that $\Phi: \mathscr{C}^{i} \to (\mathscr{C}^{-i})'$ is an isomorphism for all $|i| \geq N+1$, $N \geq 2$. Then by a similar argument we can find a hermitian complex $(\tilde{\mathscr{C}}, \tilde{\Phi})$ such that $\tilde{\Phi}: \tilde{\mathscr{C}}^{i} \to (\tilde{\mathscr{C}}^{-i})'$ is an isomorphism for $|i| \geq N$ and a homotopy equivalence $j: \mathscr{C} \to \tilde{\mathscr{C}}$ admitting a left inverse, and such that $\Phi = j'\tilde{\Phi}j$. Hence we can apply induction on N. The start of the induction is insured by the fact that $\mathscr{C}^{i} = 0$ for $|i|$ large, and the end of the induction is insured by the following remark: If the homotopy equivalence $\Phi: \hat{\mathscr{C}} \to (\hat{\mathscr{C}})'$ has the property that $\Phi: \hat{\mathscr{C}}^{i} \to (\hat{\mathscr{C}}^{-i})'$ is an isomorphism for all $|i| \geq 1$, then $\hat{\Phi}: \hat{\mathscr{C}} \to (\hat{\mathscr{C}}^{0})'$ is automatically an isomorphism. This fact is easily checked by diagram chasing. Hence the proposition is proved.

1.4. Splitting. Suppose that V is a vector bundle over Y with a nondegenerate hermitian metric (possibly indefinite). A *splitting* of V is a decomposition of V in an orthogonal sum of two sub-bundles $V = V^{+} \oplus V^{-}$ such that the metric is positive definite on V^{+} and negative definite on V^{-}. Giving a splitting of V is equivalent to a reduction of the structure group of the principal $U(p,q)$-bundle of V restricted to a connected component of Y to the subgroup $U(p) \times U(q)$. (Here $U(p,q)$ denotes the indefinite unitary group of the hermitian form $|z_1|^2 + \cdots + |z_p|^2 - |z_{p+1}|^2 - \cdots - |z_{p+q}|^2$; $U(p) \times U(q)$ is a maximal compact subgroup.) Since $U(p,q)/U(p) \times U(q)$ is homeomorphic to a euclidean space, it follows that a splitting always exists and moreover, any two splittings are homotopic.

1.5. The signature. Suppose that $(\mathscr{C}, \Phi)$ is a *regular* hermitian complex over Y. Then there is an associated invariant $\sigma(\mathscr{C}, \Phi) \in K(Y)$ called *signature*. ($K(Y)$ is the complex K-theory of Y.) This is constructed as follows: Φ defines a nondegenerte hermitian form on $\mathscr{C}^{0}$, hence one can split $\mathscr{C}^{0} = (\mathscr{C}^{0})^{+} \oplus (\mathscr{C}^{0})^{-}$ as in § 1.4. We put

$$\sigma(\mathscr{C}, \Phi) = (\mathscr{C}^{0})^{+} - (\mathscr{C}^{0})^{-} .$$

Since any two splittings of $\mathscr{C}^{0}$ are homotopic, this is independent of the choice of splitting, as an element of $K(Y)$.

1.6. Proposition. *If $(\mathscr{C}, \Phi)$ and $(\mathscr{D}, \Psi)$ are two regular hermitian complexes over Y, which are homotopy equivalent, then $\sigma(\mathscr{C}, \Phi) = \sigma(\mathscr{D}, \Psi)$.*

Proof. Let $h: \mathscr{C} \to \mathscr{D}$ be a homotopy equivalence such that Φ is homotopic to $h'\Psi h$.

Case 1. $\mathscr{C} = 0$. In this case $\mathscr{D}$ is an acyclic regular hermitian complex. It follows easily that $d(\mathscr{D}^{-1}) \subset \mathscr{D}^{0}$ is a sub-bundle of half dimension on which the hermitian form defined by Ψ is identically zero. If $\mathscr{D}^{0} = (\mathscr{D}^{0})^{+} \oplus (\mathscr{D}^{0})^{-}$ is a splitting for $\mathscr{D}^{0}$, then $(\mathscr{D}^{0})^{+}$ and $(\mathscr{D}^{0})^{-}$ are both complements for the sub-bundle $d(\mathscr{D}^{-1})$. Hence $(\mathscr{D}^{0})^{+} \approx (\mathscr{D}^{0})^{-}$ and $\sigma(\mathscr{D}, \Psi) = 0$.

Case 2. h is injective and $\Phi = h'\Psi h$. Consider the orthogonal complement $\mathscr{E}$ of $h(\mathscr{C})$ in $\mathscr{D}$ with respect to Ψ. Then $\mathscr{E}$ is an acyclic regular hermitian subcomplex of $\mathscr{D}$ and clearly $\sigma(\mathscr{D}, \Psi) = \sigma(\mathscr{C}, \Phi) + \sigma(\mathscr{E}, \Psi \mid \mathscr{E})$. But $\sigma(\mathscr{E}, \Psi \mid \mathscr{E}) = 0$ by Case 1.

Case 3. There is a homotopy equivalence $g\colon \mathscr{D} \to \mathscr{C}$ such that $gh = 1$. Then Ψ and $g'\Phi g$ are (chain-)homotopic. Hence

$$\Psi_t = t \cdot \Psi + (1 - t)g'\Phi g\ , \qquad 0 \leq t \leq 1$$

is a continuous family of homotopy-equivalences $\mathscr{D} \to \mathscr{D}'$ defining a (possibly nonregular) hermitian complex over $Y \times [0, 1]$. Applying Proposition 1.3 for the base space $Y \times [0, 1]$ we can find a continuous family of regular hermitian complexes $\hat{\mathscr{D}}_t$, $\hat{\Psi}_t$ over Y, $0 \leq t \leq 1$ and injective homotopy equivalences $f_t\colon \mathscr{D} \to \hat{\mathscr{D}}_t$ depending continuously on t and such that $\Psi_t = f_t'\hat{\Psi}_t f_t$. Then $\sigma(\hat{\mathscr{D}}_0, \hat{\Psi}_0) = \sigma(\hat{\mathscr{D}}_1, \hat{\Psi}_1)$ by continuity. On the other hand, $f_0 h\colon \mathscr{C} \to \hat{\mathscr{D}}_0$ is injective, and

$$(f_0 h)'\hat{\Psi}_0(f_0 h) = h'\Psi_0 h = h'g'\Phi g h = \Phi\ ,$$

hence $\sigma(\mathscr{C}, \Phi) = \sigma(\hat{\mathscr{D}}_0, \hat{\Psi}_0)$ by Case 2. Similarly, $f_1\colon \mathscr{D} \to \hat{\mathscr{D}}_1$ is injective, and $f_1'\hat{\Psi}_1 f_1 = \Psi_1 = \Psi$, hence $\sigma(\mathscr{D}, \Psi) = \sigma(\hat{\mathscr{D}}_1, \hat{\Psi}_1)$ by Case 2. It follows that $\sigma(\mathscr{C}, \Phi) = \sigma(\mathscr{D}, \Psi)$.

General case. First we can find a homotopy-commutative diagram

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{\ h\ } & \mathscr{D} \\ & \searrow \quad \swarrow & \\ & \mathscr{E} & \end{array}$$

where $\mathscr{E}$ is some (bounded) complex over Y and the vertical maps are homotopy equivalences admitting left-inverses. This fact is well known, or can be proved by a slightly simpler method than the proof in Proposition 1.3. On $\mathscr{E}$ we can define a natural (up to homotopy) structure θ of (possibly nonregular) hermitian complex. Using Proposition 1.3 we can find a regular hermitian complex $(\hat{\mathscr{E}}, \hat{\theta})$ and a homotopy equivalence $\mathscr{E} \to \hat{\mathscr{E}}$ admitting a left inverse and commuting with the cup-products. Applying Case 3 to the compositions $\mathscr{C} \to \hat{\mathscr{E}}$ and $\mathscr{D} \to \hat{\mathscr{E}}$ we find that $\sigma(\mathscr{C}, \Phi) = \sigma(\hat{\mathscr{E}}, \hat{\theta}) = \sigma(\mathscr{D}, \Psi)$. This concludes the proof.

1.7. Remark. As a consequence of Proposition 1.3 and Proposition 1.6, the invariant σ defined in § 1.5 only for regular hermitian complexes extends uniquely to an invariant defined for any (not necessarily regular) hermitian complex, which is a homotopy invariant of the hermitian complex.

1.8. τ-complexes. A *τ-complex* is a regular hermitian complex $(\mathscr{C}, \Phi)$ over Y together with isomorphisms $\tau\colon \mathscr{C}^i \xrightarrow{\approx} \mathscr{C}^{-i}$ defined for all i such that

(i) $\tau \circ \tau = 1$,

(ii)
$$\begin{array}{ccc} \mathscr{C}^i & \xrightarrow{\tau} & \mathscr{C}^{-i} \\ \Phi\downarrow & & \downarrow\Phi \\ (\mathscr{C}^{-i})' & \xrightarrow{\tau} & (\mathscr{C}^i)' \end{array}$$
is commutative,

(iii) for any $x \in \mathscr{C}^i$ we have $(\Phi\tau(x))(x) \geq 0$.

It is clear that the form $(x, y) \to (\Phi\tau(y))(x)$ defined on $\mathscr{C}^i \times \mathscr{C}^i$ is a positive definite hermitian form on $\mathscr{C}^i$ for any i. Also the decomposition $\mathscr{C}^0 = (\mathscr{C}^0)^+ \oplus \mathscr{C}^0)^-$ given by $(\mathscr{C}^0)^\pm = \{x \in \mathscr{C}^0 \mid \tau x = \pm x\}$ is a splitting of $(\mathscr{C}^0, \Phi)$ in the sense of § 1.4. Conversely, given a regular hermitian complex $(\mathscr{C}, \Phi)$ together with positive definite hermitian forms on $\mathscr{C}^i$, $i < 0$ and a splitting of $(\mathscr{C}^0, \Phi)$, there is a natural involution τ which makes $(\mathscr{C}, \Phi)$ a τ-complex.

1.9. Gluing together τ-complexes. We will need the following

Proposition. *Suppose given a finite covering $Y = \bigcup_{\alpha=1}^{n} Y_\alpha$ where Y_α are closed sets. For each α, $1 \leq \alpha \leq n$, there is given a τ-complex $\mathscr{C}_\alpha$ over Y_α. For each α, β, $1 \leq \alpha$, $\beta \leq n$ such that $Y_\alpha \cap Y_\beta \neq \phi$ there is given an injective homotopy equivalence over $Y_\alpha \cap Y_\beta$ either from $\mathscr{C}_\alpha$ to $\mathscr{C}_\beta$ or from $\mathscr{C}_\beta$ to $\mathscr{C}_\alpha$, commuting with τ and the cup-products, which is the identity for $\alpha = \beta$ and such that we have compatibility on sets of the form $Y_\alpha \cap Y_\beta \cap Y_\gamma$. Then there are a τ-complex $\mathscr{C}$ over Y and injective homotopy equivalences $\mathscr{C}_\alpha \to \mathscr{C}$ over Y_α, commuting with τ and the cup-products for each α, such that we have compatibility on sets of the form $Y_\alpha \cap Y_\beta$.*

Proof. We can clearly suppose that $n = 2$. The general case follows then by induction on n. Suppose that over $Y_1 \cap Y_2$ the given homotopy equivalence is $\mathscr{C}_1 \to \mathscr{C}_2$. We apply then a descending induction on N where N is such that $\mathscr{C}_1^i \to \mathscr{C}_2^i$ is isomorphism for $|i| \geq N + 1$ using the following obvious

Lemma. *Given two vector bundles V_1, V_2 over Y_1, Y_2 respectively and an inclusion $V_1 \subset V_2$ over $Y_1 \cap Y_2$, there exist vector bundles W_1, W_2 over Y_1, Y_2 respectively and an isomorphism $V_1 \oplus W_1 \xrightarrow{\approx} V_2 \oplus W_2$ defined over $Y_1 \cap Y_2$ such that the diagram*

$$\begin{array}{ccc} V_1 \oplus W_1 & \xrightarrow{\approx} & V_2 \oplus W_2 \\ \cup\uparrow & & \cup\uparrow \\ V_1 & \xrightarrow{\subset} & V_2 \end{array}$$

defined over $Y_1 \cap Y_2$ is commutative.

2. Signature with local coefficients

2.1. Flat hermitian vector bundles. Let X be a manifold of dimension $2k$, and $E \to X$ be a vector bundle with a nondegenerate hermitian metric (possibly indefinite). We suppose that a flat structure is given on E, i.e., a maximal system of trivializations of E over various open sets of X, with locally constant transition functions. Then the notion of locally constant sections of E over an open set of X is defined, and we require that the metric is flat in the sense that the inner product of two locally constant sections of E is again a locally constant functions.

Over any connected component of X, giving a vector bundle E with the structures described above, is equivalent, to a representation of the fundamental group into some $U(p, q)$ once we have chosen a base point (see § 1.4).

The flat structure defines a natural smooth structure on E, and an exterior differential $d\colon \Omega^p(X, E) \to \Omega^{p+1}(X, E)$. ($\Omega^p(X, E)$ denotes the space of smooth p-forms on X with values in E.) The metric on E defines an exterior product

$$\{\ ,\ \}\colon \Omega^p(X, E) \times \Omega^{2k-p}(X, E) \to \boldsymbol{C}$$

which is linear in the first factor and antilinear in the second. Here we have used the evaluation on the fundamental class: $\Omega^{2k}(X, \boldsymbol{C}) \to \boldsymbol{C}$.

Define: $\langle\ ,\ \rangle = \{\ ,\ \}$ on even forms, and $\sqrt{-1}\{\ ,\ \}$ on odd forms; $\tilde{d} = d$ on odd forms, and $\sqrt{-1}d$ on even forms.
Note the following formulas:

$$\tilde{d}\cdot\tilde{d} = 0\ ,$$
$$\langle\omega, \omega'\rangle = \overline{\langle\omega', \omega\rangle}\ , \qquad \deg\omega + \deg\omega' = 2k\ ,$$
$$\langle\tilde{d}\omega, \omega'\rangle = \langle\omega, \tilde{d}\omega'\rangle\ , \qquad \deg\omega + \deg\omega' + 1 = 2k\ .$$

(The absence of signs is explained by the presence of $\sqrt{-1}$ in the definition of $\langle\ ,\ \rangle$ and $\tilde{d}$.)

Denote by $H^*(X, E)$ the cohomology of the complex $(\Omega^*(X, E), \tilde{d})$. It is well known that $H^*(X, E)$ is finite dimensional. $\langle\ ,\ \rangle$ induces a nondegenerate hermitian form on $H^k(X, E)$ whose signature will be denoted by sign $(X, E) \in \boldsymbol{Z}$. Suppose that X' is another manifold of dimension $2k$, and let $f\colon X' \to X$ be a smooth map. Then the pull back bundle $f^!E$ over X' has an induced metric and flat structure, and we have a natural map $f^*\colon \Omega^*(X, E) \to \Omega^*(X', f^!E)$ commuting with $\tilde{d}$. Let f_t, $t \in [0, 1]$ be a smooth homotopy $f_t\colon X' \to X$. Define $h\colon X' \times [0, 1] \to X$ by $h(x, t) = f_t(x)$. Since the pull back bundle $h^!E$ has a canonical flat structure, we get a canonical isomorphism $h^!E = \pi^! f_0^! E$, where $\pi\colon X' \times [0, 1] \to X'$ is the projection onto the first factor. This also gives a canonical isomorphism $f_0^!E \approx f_1^!E$ which will be used to identify these two bundles together with their metrics and flat structures. We shall show that $f_0^*, f_1^*\colon \Omega^*(X, E) \to \Omega^*(X', f_0^!E)$ are chain-homotopic in a natural way. In fact, if $\omega \in \Omega^p(X, E)$, the form $h^*(\omega) \in \Omega^p(X' \times [0, 1], h^!E)$ can be written uniquely as $h^*(\omega) = j(\omega) + k(\omega)dt$, where $j(\omega) \in C^\infty([0, 1], \Omega^p(X', f_0^!E))$ and $k(\omega) \in C^\infty([0, 1], \Omega^{p-1}(X', f_0^!E))$. (We have identified here $h^!E$ to $\pi^! f_0^! E$.) Define

$$\tilde{k}(\omega) = \varepsilon_p \cdot \int_0^1 k(\omega)dt \in \Omega^{p-1}(X', f_0^! E) ,$$

where $\varepsilon_p = 1$ for odd p, and $\sqrt{-1}$ for even p. One checks easily that

$$(\tilde{d}\tilde{k} - \tilde{k}\tilde{d})(\omega) = \sqrt{-1}(-1)^{p+1}(f_1^*\omega - f_0^*\omega) ,$$

which proves our assertion.

It follows that f_0^* and f_1^* induce the same map $H^*(X, E) \to H^*(X', f_0^! E)$. If $f: X' \to X$ is a smooth homotopy equivalence, we conclude that $f^*: H^*(X, E) \to H^*(X', f^! E)$ is an isomorphism. If, moreover, f is orientation preserving, then it is compatible with the cup-product on H^k. Hence

$$\operatorname{sign}(X, E) = \operatorname{sign}(X', f^! E) .$$

2.2. The basic elliptic operator. We preserve the notations of § 2.1. We shall construct an elliptic differential operator on X whose index is equal to $\operatorname{sign}(X, E)$. This will be similar to the operator constructed in [2, § 6], except for the fact that in [2] the authors assume E to be the trivial line bundle with the standard metric and flat structure.

Choose a smooth splitting $E = E^+ \oplus E^-$ (for the definition, see § 1.4). If we change the sign of the metric on E^- and leave it unchanged on E^+, we get a positive definite hermitian metric on E which is of course not necessarily flat, because the splitting has no relation to the flat structure. Fix a riemannian metric on X. We have then induced pre-hilbert structures (,) on $\Omega^p(X, E)$. Define $\tau: \Omega^p(X, E) \to \Omega^{2k-p}(X, E)$ by the formula

$$\langle \omega, \omega' \rangle = (\omega, \tau\omega') , \qquad \deg \omega + \deg \omega' = 2k .$$

Then τ satisfies $\tau^2 = 1$, $\langle \tau\omega, \omega' \rangle = \langle \omega, \tau\omega' \rangle$, $\langle \tau\omega, \omega \rangle \geq 0$. τ defines a $C^\infty(X)$-linear involution of $\sum_p \Omega^p(X, E)$, and we define $\Omega^+(X, E)$ (resp. $\Omega^-(X, E)$) as the eigenspaces corresponding to $+1$ (resp. -1). These are the spaces of smooth sections of well-defined vector bundles over X. The adjoint of $\tilde{d}: \Omega^p(X, E) \to \Omega^{p+1}(X, E)$ with respect to the pre-hilbert structures is clearly $\tau\tilde{d}\tau: \Omega^{p+1}(X, E) \to \Omega^p(X, E)$. Then $\tilde{d} - \tau\tilde{d}\tau$ defines an elliptic operator $D: \Omega^+(X, E) \to \Omega^-(X, E)$, and one sees easily, using harmonic forms as in [2], that

$$\operatorname{index} D = \operatorname{sign}(X, E) .$$

For any $2k$-dimensional real vector bundle V consider the characteristic class

$$\tilde{\mathscr{L}}(V) = \prod_{i=1}^{k} \frac{x_i}{\tanh x_i/2} ,$$

where $p(V) = \prod_{i=1}^{k}(1 + x_i^2)$ is the total Pontrjagin class of V. Then the index theorem of [2] specializes to

$$\operatorname{sign}(X, E) = \tilde{\mathscr{L}}(X) \operatorname{ch}(E^+ - E^-)[X] .$$

(Here $\tilde{\mathscr{L}}(X)$ is $\tilde{\mathscr{L}}$ of the tangent bundle of X, and $[X]$ is the fundamental homology class of X.) Applying § 2.1 we get the following result:

Proposition. *Let X, E be as above and $X' \xrightarrow{f} X$ a smooth orientation preserving homotopy equivalence. Then*

$$\tilde{\mathscr{L}}(X) \operatorname{ch}(E^+ - E^-)[X] = \tilde{\mathscr{L}}(X') f^* \operatorname{ch}(E^+ - E^-)[X'] .$$

2.3. Remark. Suppose that $X_1 \to X_2 \to X$ is a smooth fibre bundle with X_1, X_2, X even dimensional manifolds. The middle cohomology of the fibres with complex coefficients then form a flat vector bundle H over X with a flat hermitian metric coming from the cup-product on X_1. The proofs in [1], [4] show actually that one has $\operatorname{sign}(X_2) = \operatorname{sign}(X, H)$ although the notion of signature with local coefficients is not explicity considered in these papers. Also, in [1], it is shown by an example that $\operatorname{ch}(H^+ - H^-)$ may have nonzero components in positive dimensions. Such examples are easier to find (see § 5) if we *do not* require that H comes from a geometric situation as above, but that it is an arbitrary flat hermitian bundle over X.

3. Families of flat hermitian vector bundles

In this section we shall extend the results of § 2 to families.

3.1. Let Y be a compact space, and $Z \to Y$ a locally trivial fibre bundle over Y with fibre X, a $2k$-dimensional manifold. We shall suppose that the structure group of this fibre bundle is the group $\operatorname{Diff}^+(X)$ of orientation preserving diffeomorphisms of X. Z is a *manifold over* Y in the sense of [3, § 1].

Suppose that $E \to Z$ is a vector bundle with the following two structures:

(i) a nondegenerate hermitian form (possibly indefinite),

(ii) a flat structure in the fibre direction. (This means a maximal family of trivializations of E over various open sets of Z such that the transition functions are locally constant on the intersection with any fibre X_y, $y \in Y$.)

These two structures are assumed to be compatible in the following sense: given two sections of E over some open set U in Z, which are "locally constant" on any intersection $U \cap X_y$, $y \in Y$, their inner product must be locally constant on any $U \cap X_y$, $y \in Y$.

Then for any $y \in Y$, the restriction bundle $E_y = E|X_y$ is of the type described in § 2.1; in particular it has a natural smooth structure.

Let $y^0 \in Y$, and Y^0 be an open neighbourhood of y^0 such that $Z|Y^0 \approx X \times Y^0$ and $E|(Z|Y^0) \approx E_{y^0} \times Y^0$. Let X^0 be a connected component of X, Z^0 the subset of $Z \mid Y^0$ corresponding to $X^0 \times Y^0$, and $x^0 \in X^0$. Then the bundle $\boldsymbol{E} \mid Z^0$ and its two structures can also be described up to isomorphism by a continuous family of representations ρ_y, $y \in Y^0$ of the fundamental group $\pi_1(X^0, x^0)$ into $U(p, q)$. (This means by definition that for any $g \in \pi_1(X^0, x^0)$, the map $Y \rightarrow U(p, q)$ defined by $y \rightarrow \rho_y(g)$ is continuous.) The system $(X \rightarrow Y \rightarrow Z, \boldsymbol{E})$ will be called a family of flat hermitian vector bundles.

Define $\tilde{\Omega}^i = \bigcup_{y \in Y} \tilde{\Omega}^i_y$ where $\tilde{\Omega}^i_y = \Omega^{i+k}(X_y, \boldsymbol{E}_y)$.[2] We can make $\tilde{\Omega}^i$ into a bundle of Fréchet spaces over Y in the following way: Any trivialization $\boldsymbol{E} \mid (Z \mid Y_0) \approx \boldsymbol{E}_{y_0} \times Y^0$ over some open neighbourhood Y^0 of y_0 (which is smooth and linear in the fibre direction but in general not compatible with the flat structures and metrics) defines a bijection $\tilde{\Omega}^i \mid Y_0 \approx \tilde{\Omega}^i_{y_0} \times Y_0$ which is linear on fibres and induces a topology on $\tilde{\Omega}^i \mid Y_0$. The topologies obtained on subsets of $\tilde{\Omega}^i$ for various trivializations of $\boldsymbol{E}$ are compatible, and define a topology on $\tilde{\Omega}^i$.

The bundle maps $\tilde{d}: \tilde{\Omega}^i \rightarrow \tilde{\Omega}^{i+1}$ define then a complex $\tilde{\Omega}$ over Y. The pairing $\langle\, ,\rangle$ considered in § 2.1 can be considered for each fibre X_y, and defines a separately continuous pairing $\langle\, ,\rangle$: $\tilde{\Omega}^i \times \tilde{\Omega}^{-i} \rightarrow 1$ where 1 denotes the trivial line bundle over Y. Choose a splitting $\boldsymbol{E} = \boldsymbol{E}^+ \oplus \boldsymbol{E}^-$, which is smooth in the fibre-direction, and a riemannian metric on the tangent space along the fibres $T(Z/Y)$ smooth in the fibre-direction. Then, as in § 2.1 we find induced pre-hilbert structures on the bundles $\tilde{\Omega}^i$ and bundle involutions $\tau: \tilde{\Omega}^i \rightarrow \tilde{\Omega}^{-i}$ satisfying the relations of § 2.1. We define $\tilde{\Omega}^{\pm}$ as the ± 1-eigenspaces of τ acting in $\sum_i \tilde{\Omega}^i$, and we have a *family of elliptic differential operators* $D = \tilde{d} - \tau\tilde{d}\tau$: $\tilde{\Omega}^+ \rightarrow \tilde{\Omega}^-$ in the sense of [3, § 1].

Let $\operatorname{sign}(Z/Y; \boldsymbol{E}) \in K(Y)$ denote the analytical index of D. We shall show in § 3.2 that $\operatorname{sign}(Z/Y; \boldsymbol{E})$ is in fact the signature of some hermitian complex constructed out of the eigenspaces of the Laplace operator.

We recall the definition of the analytical index, see [3, § 2], [15], in a form convenient to us. Suppose we have found two finite dimensional vector bundles V and W over Y and morphisms

$$\begin{array}{ccc} \tilde{\Omega}^+ & \xrightarrow{D} & \tilde{\Omega}^- \\ \uparrow \phi & & \uparrow \psi \\ V & \xrightarrow{\delta} & W \end{array}$$

such that $\psi\delta = D\phi$ and such that ϕ: $\ker \delta \xrightarrow{\approx} \ker D$, ψ: $\operatorname{coker} \delta \xrightarrow{\approx} \operatorname{coker} D$. Then, by definition,

$$\operatorname{sign}(Z/Y; \boldsymbol{E}) = V - W \in K(Y)\ .$$

[2] The shift in dimension is motivated by the notation of §1.

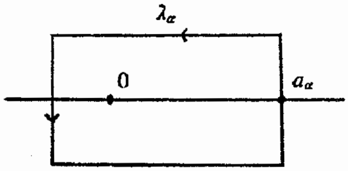
One can show that this definition is independent of the choices made. It is also clear that sign $(Z/Y; E)$ is independent of the chosen splitting and riemannian metric.

3.2. Our next task will be to exhibit a particular choice for V and W mentioned at the end of § 3.1. Observe that $\tilde{\Omega}$ together with the pairing $\langle\,,\rangle$ and the involution is very much like a τ-complex (see § 1.8) except for the fact that it has infinite-dimensional bundles. We will construct now a genuine τ-complex associated to $\tilde{\Omega}$. Let Δ_y^i be the Laplace operator in $\tilde{\Omega}_y^i$ defined by

$$\Delta_y^i = \tilde{d}\tau\tilde{d}\tau + \tau\tilde{d}\tau\tilde{d}\ , \qquad y \in Y\ .$$

Let $a_y > 0$ such that a_y is not an eigenvalue for Δ_y^i for any i. By continuity of the eigenvalues, a_y is not an eigenvalue for $\Delta_{y'}^i$ for any i and $y' \in N_y$, a closed neighbourhood of y in Y. Choose a finite number of points $y_1, \cdots, y_n \in Y$ such that the interiors of $Y_\alpha = N_{y_\alpha}$ cover Y, and put $a_\alpha = a_{y_\alpha}$.

Consider the following path λ_α in $\boldsymbol{C}$:



(λ_α is a rectangle which crosses the real axis at a_α and at some negative point.)

The operators

$$A_{\alpha,y}^i = \int_{\lambda_\alpha} (\Delta_\alpha^i - \zeta)^{-1} d\zeta\ , \qquad B_{\alpha,y}^i = \int_{\lambda_\alpha} (\Delta_y^i - \zeta)^{-1} \frac{d\zeta}{\zeta}$$

are well defined in $\tilde{\Omega}_y^i$ for $y \in Y_\alpha$ and depend continuously on y. From the Cauchy integral formula and the spectral decomposition of Δ_y^i, it follows that

$$\begin{aligned}\mathscr{C}_{\alpha,y}^i &= \text{Image } A_{\alpha,y}^i = \ker B_{\alpha,y}^i \\ &= \text{subspace of } \tilde{\Omega}_y^i \text{ spanned by the eigenvectors of } \Delta_y^i \\ &\quad\ \text{corresponding to eigenvalues less than } a_\alpha .\end{aligned}$$

Then $\dim \mathscr{C}_{\alpha,y}^i$ is finite and both upper and lower semi-continuous in y, so that it is locally constant with respect to y. Hence $\mathscr{C}_\alpha^i = \bigcup_{y \in Y_\alpha} \mathscr{C}_{\alpha,y}^i$ is a (finite dimensional) subvector bundle of $\tilde{\Omega}^i | Y_\alpha$. Clearly, the Laplacian commutes with $\tilde{d}$ and τ, hence also $A_{\alpha,y}^*$, $B_{\alpha,y}^*$ do. It follows that $\tilde{d}(\mathscr{C}_\alpha^i) \subset \mathscr{C}_\alpha^{i+1}$ and $\tau(\mathscr{C}_\alpha^i) = \mathscr{C}_\alpha^{-i}$. The complex of vector bundles $\mathscr{C}_\alpha = \left\{\cdots \to \mathscr{C}_\alpha^i \xrightarrow{\tilde{d}} \mathscr{C}_\alpha^{i+1} \to \cdots\right\}$ together with the involution τ and the cup-product $\langle\ ,\ \rangle$ is then a τ-complex over Y_α.

Denoted by $p_\alpha\colon \tilde{\Omega} \to \mathscr{C}_\alpha$ and $q_\alpha\colon \mathscr{C}_\alpha \to \tilde{\Omega}$ the canonical orthogonal projection and inclusion, respectively. Then p_α and q_α are homotopy equivalences commuting with τ. Moreover q_α commutes also with the cup-product, and $p_\alpha q_\alpha = 1$. Suppose now that $Y_\alpha \cap Y_\beta \neq \phi$. We have $a_\alpha \leq a_\beta$ or $a_\alpha \geq a_\beta$. Suppose the first inequality is satisfied. Then clearly over $Y_\alpha \cap Y_\beta$ we have a canonical imbedding $\mathscr{C}_\alpha \xrightarrow{h_{\alpha\beta}} \mathscr{C}_\beta$ which is a homotopy equivalence commuting with τ and the cup-products. (In fact, the orthogonal complement of $\mathscr{C}_\alpha$ in $\mathscr{C}_\beta$ is acyclic because it is a direct sum of eigenspaces of the Laplacian corresponding to nonzero eigenvalues.)

We can apply § 1.9 to this situation: we find a τ-complex $\mathscr{C}$ over Y and injective homotopy equivalences $\mathscr{C}_\alpha \xrightarrow{h_\alpha} \mathscr{C}$ over Y_α, commuting with τ and the cup-products and such that over $Y_\alpha \cap Y_\beta$ we have $h_\alpha = h_\beta h_{\alpha\beta}$ or $h_\beta = h_\alpha h_{\beta\alpha}$ (depending on which of the inequalities $a_\alpha \leq a_\beta$ or $a_\beta \leq a_\alpha$ hold). Let r_α be the orthogonal projection of $\mathscr{C}$ onto $\mathscr{C}_\alpha$. Consider a partition of unity $\{\phi_\alpha\}_{1\leq\alpha\leq n}$ such that $\operatorname{supp}\phi_\alpha \subset \text{Interior}\,(Y_\alpha)$.

The maps

$$p = \sum_\alpha \phi_\alpha h_\alpha p_\alpha\colon \tilde{\Omega} \to \mathscr{C}\,, \qquad q = \sum_\alpha \phi_\alpha h_\alpha r_\alpha\colon \mathscr{C} \to \tilde{\Omega}$$

are homotopy equivalences defined over Y, homotopy inverses to each other, commuting with τ; for any fixed $y \in Y$, the induced maps in the cohomology over y commute with the cup-products.

Denote by $\mathscr{C}^\pm$ the $\pm$-eigenspaces of τ acting in $\sum \mathscr{C}^i$, and by $d_\mathscr{C}$ the differential of $\mathscr{C}$. Consider the commutative diagram:

$$\begin{array}{ccc} \Omega^+ & \xrightarrow{\ D\ } & \Omega^- \\ {\scriptstyle q}\uparrow & & \uparrow{\scriptstyle q} \\ \mathscr{C}^+ & \xrightarrow[d_\mathscr{C} - \tau d_\mathscr{C} \tau]{} & \mathscr{C}^- \end{array}$$

Then q induces isomorphisms between the kernels of the two horizontal lines and also between their cokernels. It follows (see the end of § 3.1) that

$$\operatorname{sign}(Z/Y;\, E) = \mathscr{C}^+ - \mathscr{C}^- \in K(Y)\ .$$

It is easy to see that $\mathscr{C}^+ - \mathscr{C}^- = (\mathscr{C}^0)^+ - (\mathscr{C}^0)^-$. ($\mathscr{C}^0 \cap \mathscr{C}^+$ and $\mathscr{C}^0 \cap \mathscr{C}^-$ define a splitting of $\mathscr{C}^0$.) Hence $\operatorname{sign}(Z/Y;\, E) = \sigma(\mathscr{C})$, where $\sigma(\mathscr{C})$ is the signature (see § 1.5) of the τ-complex $\mathscr{C}$, and we have proved the following

Proposition. *Let $X \to Z \to Y$, E, $\tilde{\Omega}$ be as in* § 3.1. *Then there exist a (finite dimensional) regular hermitian complex $\mathscr{C}$ over Y and homotopy equivalences $p\colon \tilde{\Omega} \to \mathscr{C}$, $q\colon \mathscr{C} \to \tilde{\Omega}$ homotopy inverses to each other, commuting on* *cohomology level with the cup-products for any fixed* $y \in Y$ *and such that*

$$\operatorname{sign}(Z/Y;\ E) = \sigma(\mathscr{C}) .$$

Remark. This proposition should be compared with a result of G. Segal [15] asserting that a family of elliptic complexes over a compact space Y has the homotopy type of a finite dimensional complex over Y. Actually, this result follows also from our methods.

3.3. Theorem. *Let* Y, Y' *be two compact spaces and* $X \to Z \xrightarrow{\pi} Y$, $X' \to Z' \xrightarrow{\pi'} Y'$ *2k-manifolds over* Y, Y' *as defined in* § 2.1. *Let* $E \to Z$ *be a family of flat hermitian bundles as in* § 3.1. *Suppose that* $g\colon Y' \to Y$ *is a homeomorphism and that* $f\colon Z' \to Z$ *is a smooth*[3] *orientation preserving fibre-homotopy equivalence lying over* g. *Consider the induced family of flat hermitian bundles* $f^! E$ *over* Z'. *Then*

$$\operatorname{sign}(Z'/Y';\ f^! E) = g^!(\operatorname{sign}(Z/Y;\ E)) \in K(Y') .$$

Proof. We can suppose that $Y = Y'$ and g is the identity. Let $\tilde{\Omega}$, $'\tilde{\Omega}$ be the complexes of differential forms along the fibres associated to Z, Z', and $\mathscr{C} \underset{p}{\overset{q}{\rightleftarrows}} \tilde{\Omega}$ $'\mathscr{C} \underset{'p}{\overset{'q}{\rightleftarrows}} '\tilde{\Omega}$ be[4] as in Proposition 3.2. f induces a homotopy equivalence $f^*\colon \tilde{\Omega} \to '\tilde{\Omega}$. This is proved by the explicit homotopy constructed in § 2.1. Then $'pf^*q\colon \mathscr{C} \to '\mathscr{C}$ is a homotopy equivalence which commutes up to homotopy with the cup-products. By Proposition 1.6 we have $\sigma(\mathscr{C}) = \sigma('\mathscr{C})$ and now the theorem follows from § 3.2.

The following is a generalization of § 2.2 (which corresponds to the case $Y =$ point):

Corollary. *Under the hypothesis of Theorem* 3.3 *one has*

$$\pi'_*(\tilde{\mathscr{L}}(Z'/Y') f^* \operatorname{ch}(E^+ - E^-)) = g^*\pi_*(\tilde{\mathscr{L}}(Z/Y) \operatorname{ch}(E^+ - E^-)) \in H^{ev}(Y; Q) .$$

(π_* denotes the integration along fibres: $H^*(Z; Q) \to H^*(Y; Q)$ which decreases degrees by $2k$; similarly for π'_*. We use Čech cohomology. By definition, $\tilde{\mathscr{L}}(Z/Y) = \tilde{\mathscr{L}}(T(Z/Y)$; similarly for $\tilde{\mathscr{L}}(Z'/Y')$.)

This follows by applying the Chern character to the equality of the Theorem and using the Atiyah-Singer index theorem for families of elliptic operators [3, Theorem 5.1].

[3] "smooth" means: smooth in the fibre direction.
[4] Caution: $'\mathscr{C}$ should not be confused with $\mathscr{C}'$, the antidual of $\mathscr{C}$.

4. Novikov's higher signature

In this section we specialize the results of § 3 to more concrete geometric situations. In particular we show how Novikov's higher signature comes from a family of flat hermitian bundles over some torus.

4.1. Duality of tori. By definition, a torus is a compact connected abelian Lie group. If T is a torus, the Pontrjagin dual $\hat{T} = \mathrm{Hom}_{\mathrm{cont}}(T, S^1)$ is a finitely generated free abelian group with discrete topology. Here $S^1 = \boldsymbol{R}/\boldsymbol{Z}$. The Pontrjagin dual of $\mathrm{Hom}_{\boldsymbol{Z}}(\hat{T}, \boldsymbol{Z})$ is then again a torus T^* which will be called the torus *dual* to T. Let $\Gamma = (T^*)^\wedge$ and $V = \Gamma \otimes_{\boldsymbol{Z}} \boldsymbol{R}$. Then T is canonically isomorphic to V/Γ.

Similarly, if we put $\Gamma^* = \mathrm{Hom}_{\boldsymbol{Z}}(\Gamma, \boldsymbol{Z})$ and $V^* = \Gamma^* \otimes_{\boldsymbol{Z}} \boldsymbol{R} = \mathrm{Hom}_{\boldsymbol{R}}(V, \boldsymbol{R})$, then T^* is canonically isomorphic to V^*/Γ^*. Let $\langle\, ,\, \rangle$ be the duality pairing $V \times V^* \to \boldsymbol{R}$. Consider the action of $\Gamma \times \Gamma^*$ on $V \times V^* \times \boldsymbol{C}$ defined by

$$(v, v^*, \lambda), (\gamma, \gamma^*) \to (v + \gamma, v^* + \gamma^*, \exp(2\pi i\langle \gamma, v^* \rangle)\cdot \gamma) .$$

Here $\gamma \in \Gamma$, $\gamma^* \in \Gamma^*$, $v \in V$, $v^* \in V^*$, $\gamma \in \boldsymbol{C}$. The orbit space $(V \times V^* \times \boldsymbol{C})/(\Gamma \times \Gamma^*)$ is then a smooth complex line bundle L over the torus

$$(V \times V^*)/(\Gamma \times \Gamma^*) = T \times T^* ,$$

the projection being given by $(v, v^*, \lambda) \to (v, v^*)$.

We want to determine the Chern class

$$c_1(L) \in H^2(T \times T^*; \boldsymbol{Z}) .$$

Observe that there are canonical isomorphisms $H^1(T; \boldsymbol{Z}) = \Gamma^*$, $H^1(T^*; \boldsymbol{Z}) = \Gamma$ and that there is a canonical element

$$\alpha \in \Gamma^* \otimes_{\boldsymbol{Z}} \Gamma = H^1(T; \boldsymbol{Z}) \otimes H^1(T^*; \boldsymbol{Z}) \subset H^2(T \times T^*; \boldsymbol{Z})$$

corresponding to the identity homomorphism $\Gamma \to \Gamma$ by the isomorphism $\Gamma^* \otimes_{\boldsymbol{Z}} \Gamma = \mathrm{Hom}_{\boldsymbol{Z}}(\Gamma, \Gamma)$. (The inclusion $\subset$ is given by the cup-product.)

Proposition. $c_1(L) = \alpha$.

Proof. Let $f\colon V \times V^* \to \boldsymbol{C}$ be a smooth function and $(w, w^*) \in V \times V^*$. Define a new smooth function $\nabla_{(w,w^*)}f$ on $V \times V^*$ by

$$(\nabla_{(w,w^*)}f)(v, v^*) = \frac{\partial f}{\partial w}(v, v^*) + \frac{\partial f}{\partial w^*}(v, v^*) - 2\pi i\langle v, w^* \rangle f(v, v^*) .$$

One checks easily that $\nabla_{(w,w^*)}$ leaves invariant the subspace

$$\{f \mid f(v + \gamma, v^* + \gamma^*) = \exp(2\pi i\langle \gamma, v^* \rangle)f(v, v^*) ,$$
$$\forall (v, v^*, \gamma, \gamma^*) \in V \times V^* \times \Gamma \times \Gamma^*\} \subset C^\infty(V \times V^*; \boldsymbol{C}) ,$$

which can be identified with the space of smooth sections of L. Hence L has a *natural connection* ∇. (Remark: (w, w^*) can be considered as a tangent vector at an arbitrary point of $T \times T^*$, the latter being parallelizable in a natural way.)

The curvature K of the connection ∇ is given by the formula:

$$K_{(u,u^*),(w,w^*)}f = \nabla_{(u,u_*)}\nabla_{(w,w^*)}f - \nabla_{(w,w_*)}\nabla_{(u,u^*)}f$$
$$= 2\pi i(-\langle u, w^*\rangle + \langle w, u^*\rangle)f\,,$$

where $u, w \in V$, $u^*, w^* \in V^*$.

It is well known that as a real differential form, $c_1(L)$ is represented by $iK/(2\pi)$. From the above explicit formula for K, the proposition follows easily.

Remark. The line bundle L restricted to any fibre of the projection $T \times T^* \to T^*$ has a natural flat structure and a flat, positive definite, hermitian metric. In fact, if we fix $v^* \in V^*$, then the restriction of L to $T \times \{v^* + \Gamma^*\} \subset T \times T^*$ is obtained from the unitary representation of the fundamental group Γ into S^1, given by $\gamma \to \exp(2\pi i\langle\gamma, v^*\rangle)$. Clearly, this flat structure varies continuously (see § 3.1) as v^* varies.

4.2. Homotopy invariance of Novikov's higher signature. Consider a smooth map $X \xrightarrow{\rho} T$, where X is a $2k$-manifold and T is a torus. Let T^* be the dual torus. Then, as explained in § 4.1, we have a natural line bundle L over $T \times T^*$ which has a flat structure and a flat hermitian metric when restricted to any fibre of $T \times T^* \to T^*$.

Let $(\rho \times 1)^! L$ be the line bundle over $X \times T^*$ induced by the map $X \times T^* \xrightarrow{\rho \times 1} T \times T^*$. Then the system $(X \to X \times T^* \to T^*, (\rho \times 1)^! L)$ is a family of flat hermitian bundles (see § 3.1). Put (see § 3.1)

$$\operatorname{sign}_\rho(X) = \operatorname{sign}(X \times T^*/T^*, (\rho \times 1)^! L) \in K(T^*)\,.$$

Theorem. *Let X, X' be $2k$-manifolds, T, T' two tori and*

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ {\scriptstyle\rho'}\downarrow & & \downarrow{\scriptstyle\rho} \\ T' & \xrightarrow{g} & T \end{array}$$

a homotopy-commutative diagram of smooth maps such that f is an orientation preserving homotopy equivalence and g an ismorphism of Lie groups. Let g^* denote the dual isomorphism of the dual tori: $T^* \to (T')^*$. Then $(g^*)^! \operatorname{sign}_{\rho'}(X') = \operatorname{sign}_\rho(X)$.

Proof. Consider the fibre-homotopy commutative diagram:

$$\begin{array}{ccc} X' \times (T')^* & \xrightarrow{f \times (g^*)^{-1}} & X \times T^* \\ \downarrow & & \downarrow \\ (T')^* & \xrightarrow{(g^*)^{-1}} & T^* \end{array}$$

We have a line bundle $(\rho \times 1)^! L$ on $X \times T^*$ and similarly a line bundle $(\rho' \times 1)^! L'$ on $X' \times (T')^*$. Since L and L' were constructed in a functorial way, we have $L' = (g \times (g^*)^{-1})^! L$. Since L is flat in the fibre-directions, a fibre-homotopy between $(g \times (g^*)^{-1})(\rho' \times 1)$ and $(\rho \times 1)(f \times (g^*)^{-1})$ defines an idetification of $(\rho' \times 1)^!(g \times (g^*)^{-1})^! L$ and $(f \times (g^*)^{-1})^!(\rho \times 1)^! L$. Hence $(\rho' \times 1)^! L'$ and $(f \times (g^*)^{-1})^!(\rho \times 1)^! L$ can be identified (together with their flat structures along fibres and metrics). The theorem follows directly from Theorem 3.3.

We shall apply this theorem in the case $T = H_1(X; S^1)$, considered as a Lie group. The dual torus T^* can be naturally identified with $H^1(X; S^1)^0$. (The superscript ^{0} denotes the component of the identity.) We shall denote $T =$ Alb $(X), T^* =$ Pic (X) and call them, in analogy with algebraic geometry, the Albanese, resp. Picard torus of X. Suppose X to be connected[5], and choose a base point $x_0 \in X$ and a Riemann metric on X. These data define a smooth map $X \xrightarrow{\rho}$ Alb (X) by the formula

$$x \in X \to \left\{ \omega \in H^1(X; \boldsymbol{R}) \to \int_{x_0}^{x} \omega \right\} ,$$

which should be interpreted in the following way: given $x \in X$, choose a smooth path γ from x_0 to x. Any $\omega \in H^1(X; \boldsymbol{R})$ can be represented by a unique harmonic 1-form on X whose integral along γ is denoted by $\int_{x_0}^{x} \omega$. Thus we get a linear form on $H^1(X; \boldsymbol{R})$, i.e., an element in $H_1(X; \boldsymbol{R})$. If we change the path γ in some other path γ' from x_0 to x, our element is modified with the element of $H_1(X, \boldsymbol{Z})$ determined by the closed path $(\gamma')^{-1} \circ \gamma$. We hence have a well defined element $\rho(x) \in H_1(X; \boldsymbol{R})/\text{image } H_1(X; \boldsymbol{Z}) =$ Alb (X). Clearly, the homotopy class of ρ is independent of the choice of the base point and riemannian metric. (Actually, the functors Alb (X) and Pic (X) are defined for any polyhedron X with finite 1-skeleton, and there is always a canonical map up to homotopy $\rho\colon X \to$ Alb (X); ρ is characterized by the property that if $H_1(\text{Alb}(X); \boldsymbol{Z})$ is identified with $H_1(X, \boldsymbol{Z})/\text{Tors}$, then $H_1(\rho)\colon H_1(X; \boldsymbol{Z}) \to H_1(X; \boldsymbol{Z})/\text{Tors}$ is the natural projection.)

For any $2k$-manifold X we define the "Novikov higher signature" by the formula

$$\text{Nov}(X) = \text{sign}_\rho (X) \in K(\text{Pic}(X)) ,$$

where $\rho\colon X \to \text{Alb}(X)$ is the canonical map. From the above theorem we deduce the following

Corollary. *Let $X' \xrightarrow{f} X$ be a smooth orientation preserving homotopy*

[5] This assumption can easily be removed.

equivalence of 2k-manifolds. Identify $K(\text{Pic}(X)) = K(\text{Pic}(X'))$ *via f. Then*

$$\text{Nov}(X) = \text{Nov}(X') \in K(\text{Pic}(X)) .$$

Actually, the theorem can be deduced from the corollary, using the following property of the map $X \xrightarrow{\rho} \text{Alb}(X)$: Given any map $X \xrightarrow{\rho_1} T$ into some torus there exists a unique homomorphism of tori $\text{Alb}(X) \xrightarrow{h} T$ such that $\rho_1 = h \circ \rho$.

4.3. Poincaré complexes. The homotopy invariance theorem proved in § 4.2 suggests that Nov (X) might be defined purely in terms of the Poincaré duality on X. We will show that this is, indeed, the case.

Suppose that X is a finite polyhedron. Assume that X satisfies Poincaré duality in the following form: A fundamental class $[X] \in H_{2k}(X; Z)$ is given, and for any local coefficient system S of complex vector spaces of dimension 1 over X, the map $H^i(X; S) \xrightarrow{\cap [X]} H_{2k-i}(X; S)$ is assumed to be an isomorphism for any i. Such an X will be called a *Poincaré complex* of formal dimension $2k$. Let $\rho: X \to T$ be a continuous map of X into some torus. Then we have a natural line bundle L over $T \times T^*$ as explained in § 4.1. For any $t^* \in T^*$ let $\mathcal{S}^i_{t^*}$ be the space of simplicial i-cochains of X with values in the flat bundle (or local coefficient system) $(\rho \times 1)^! L | X \times \{t^*\}$, where $\rho \times 1: X \times T^* \to T \times T^*$. Then $\mathcal{S}^i = \bigcup_{t^* \in T^*} \mathcal{S}^i_{t^*}$ is in a natural way a (finite dimentional) vector bundle over T^*. The set of usual coboundary operators $\mathcal{S}^i_{t^*} \to \mathcal{S}^{i+1}_{t^*}$ for all t^* define morphisms $d: \mathcal{S}^i \to \mathcal{S}^{i+1}$ such that $d \circ d = 0$. Using the hermitian metric in the bundle $(\rho \times 1)^! L | X \times \{t^*\}$ we get a pairing $\mathcal{S}^i_{t^*} \times \mathcal{S}^{2k-i}_{t^*} \to C$ $(0 \leq i \leq 2k)$, given by the usual explicit formula for cup-products, followed by evaluation on a chain-representative of $[X]$ fixed once for all. The set of these pairings for all t^* defines a pairing $\{\, ,\}: \mathcal{S}^i \times \mathcal{S}^{2k-i} \to 1$ (1 is the trivial line bundle ovdr T^*) which is linear in the first variable and antilinear in the second. Put

$$\tilde{\mathcal{S}}^i = \begin{cases} \mathcal{S}^{i+k} , & \text{for } -\infty < i < k , \\ \ker (\mathcal{S}^{2k} \xrightarrow{d} \mathcal{S}^{2k+1}) , & \text{for } i = k , \\ 0 , & \text{for } i > k . \end{cases}$$

Note that $\ker (\mathcal{S}^{2k} \xrightarrow{d} \mathcal{S}^{2k+1})$ is a vector bundle since the complex $(\mathcal{S}, d)$ is acyclic in degrees $\geq 2k + 1$. Define $\tilde{d}: \tilde{\mathcal{S}}^i \to \tilde{\mathcal{S}}^{i+1}$ by $\tilde{d} = d$ for odd i, and $\sqrt{-1}d$ for even i, and define $\langle\, ,\rangle_0: \tilde{\mathcal{S}}^i \times \tilde{\mathcal{S}}^{-i} \to 1$ by $\langle\, ,\rangle_0 = \{\, ,\}$ for even i, and $\sqrt{-1}\{\, ,\}$ for odd i (compare § 2.1). The pairing $\langle\, ,\rangle_0$ is in general not symmetric since the cupproduct is not symmetric on cochain level. We remedy this by defining

$$\langle x_1, x_2 \rangle = \tfrac{1}{2}[\langle x_1, x_2 \rangle_0 + \langle x_2, x_1 \rangle_0]$$

for $x_1 \in \tilde{\mathscr{S}}^i$, $x_2 \in \tilde{\mathscr{S}}^{-i}$. It follows from Poincaré duality and § 1.2 that $(\tilde{\mathscr{S}}, \tilde{d}, \langle\ \rangle)$ is a (possibly nonregular) hermitian complex over T^*. Elementary simplicial cohomology theory shows that the homotopy type of this hermitian complex (cf. § 1.2) depends only on the homotopy type of X together with the homology class $[X]$ and a homotopy class of maps $X \to T$. Hence the signature $\sigma(\tilde{\mathscr{S}}) \in K(T^*)$ is a homotopy invariant of the situation (cf. Remark 1.7).

Let us compare $\sigma(\tilde{\mathscr{S}})$ with the invariant defined in § 4.2. Suppose that X is a $2k$-manifold together with a smooth map $X \xrightarrow{\rho} T$. Consider some triangulation of X. Let $\tilde{\Omega}$ be the complex of differential forms along the fibres of $X \times T^* \to T^*$ with values in $(\rho \times 1)^! L$. ($\tilde{\Omega}$ is an infinite-dimensional vector bundle over T^*; see § 3.1.) The process of integration of an i-form on an i-simplex defines a morphism $h\colon \tilde{\Omega} \to \tilde{\mathscr{S}}$ of complexes over T^*, and it follows, from de Rham's theorem with local coefficients, that for any $t^* \in T^*$ the morphism $\tilde{\Omega}_{t^*} \to \tilde{\mathscr{S}}_{t^*}$ induces an isomorphism in cohomology compatible with the cup-products. Let $\mathscr{C} \underset{p}{\overset{q}{\rightleftarrows}} \tilde{\Omega}$ be as in § 3.2. Then $hq\colon \mathscr{C} \to \tilde{\mathscr{S}}$ is an isomorphism in cohomology for any fixed t^*. It follows that hq is a homotopy equivalence of hermitian complexes (see Lemma 1.2). By Remark 1.7, $\sigma(\mathscr{C}) = \sigma(\tilde{\mathscr{S}})$. But from § 3.2 we have $\sigma(\mathscr{C}) = \operatorname{sign}_\rho(X)$. Hence $\sigma(\tilde{\mathscr{S}}) = \operatorname{sign}_\rho(X)$. We shall summarize the above discussion specializing to the case where $T = \operatorname{Alb}(X)$, $T^* = \operatorname{Pic}(X)$, and $\rho\colon X \to T$ is the canonical map.

Theorem. *Let X be any Poincaré complex of formal dimension $2k$. Then there is an invariant* $\operatorname{Nov}(X) \in K(\operatorname{Pic}(X))$ *with the following properties*:

(i) *If $f\colon X' \to X$ is a homotopy equivalence of Poincaré complexes such that $f_*[X'] = [X]$, then* $\operatorname{Nov}(X)$ *and* $\operatorname{Nov}(X')$ *correspond to each other by the identification* $K(\operatorname{Pic}(X)) = K(\operatorname{Pic}(X'))$ *induced by f.*

(ii) *If X is a $2k$-manifold, then* $\operatorname{Nov}(X)$ *is the same as the invariant defined in* § 4.2.

4.4. Cohomological expression of $\operatorname{Nov}(X)$. Let X be again a $2k$-manifold, and $\operatorname{Nov}(X) \in K(\operatorname{Pic}(X))$ its Novikov higher signature (cf. § 4.2). Since $\operatorname{Nov}(X)$ is the index of a family of elliptic operators, its Chern character can be calculated as in the corollary in § 3.3. We have

$$\operatorname{ch}\operatorname{Nov}(X) = \mathscr{L}(X)\operatorname{ch}(\rho \times 1)^! L[X] \in H^{ev}(\operatorname{Pic}(X); \boldsymbol{Z}),$$

where $\rho\colon X \to \operatorname{Alb}(X)$ is the canonical map, and $L \to \operatorname{Alb}(X) \times \operatorname{Pic}(X)$ is the canonical line bundle defined in § 4.1. Let $x_1, \cdots, x_n$ be a basis of $H^1(X; \boldsymbol{Z}) = H^1(\operatorname{Alb}(X); \boldsymbol{Z})$, and $\xi_1, \cdots, \xi_n$ the dual basis of $H_1(X; \boldsymbol{Z})/\mathrm{Tors} = H^1(\operatorname{Pic}(X); \boldsymbol{Z})$. By the proposition in § 4.1, $c_1(L) = \sum_{i=1}^{n} x_i \xi_i$, from which we deduce

$$c_1^r(L) = (-1)^{r(r-1)/2} r! \sum_{1 \le i_1 < \cdots < i_r \le n} x_{i_1} \cdots x_{i_r} \xi_{i_1} \cdots \xi_{i_r}, \qquad r \ge 0,$$

$$\operatorname{ch}(L) = \exp(c_1(L)) = \sum_{r\geq 0}(-1)^{r(r-1)/2} \sum_{1\leq i_1<\cdots<i_r\leq n} x_{i_1}\cdots x_{i_r}\xi_{i_1}\cdots\xi_{i_r} .$$

Identifying $x_i = \rho^* x_i$ and noting that $\frac{1}{2}r(r-1) \equiv k (\bmod 2)$, if $r \equiv 2k (\bmod 4)$ we have:

$$(*) \qquad \begin{aligned} &\operatorname{ch}\operatorname{Nov}(X) \\ &\quad = (-1)^k \sum_{\substack{0\leq r\leq n \\ r\equiv 2k(4)}} \sum_{0\leq i_1<\cdots<i_r\leq n} \tilde{\mathscr{L}}_{2k-r}(X)x_{i_1}\cdots x_{i_r}[X]\cdot\xi_{i_1}\cdots\xi_{i_r} .\end{aligned}$$

If $M = M_{i_1\cdots i_r}$ denotes an oriented submanifold of X with trivial normal bundle dual to $x_{i_1}\cdots x_{i_r}$ (this is of course empty if $r > 2k$), then the Hirzebruch signature theorem applied to M gives

$$\begin{aligned}\operatorname{sign}(M_{i_1\cdots i_r}) &= \tilde{\mathscr{L}}_{2k-r}(M)[M] = \frac{i^*\tilde{\mathscr{L}}_{2k-r}(X)}{\mathscr{L}_0(1 \underbrace{\oplus \cdots \oplus}_{r} 1)}[M] \\ &= 2^{-r/2} i^*\tilde{\mathscr{L}}_{2k-r}(X)[M] = 2^{-r/2}\tilde{\mathscr{L}}_{2k-r}(X)x_{i_1}\cdots x_{i_r}[X] .\end{aligned}$$

Here i is the inclusion $M \subset X$ and $r \equiv 2k \pmod 4$. Substituting in (*) we obtain

$$\begin{aligned}\operatorname{ch}\operatorname{Nov}(X) &= (-1)^k \sum_{\substack{0\leq r\leq n \\ r\equiv k2(4)}} \sum_{0\leq i_1<\cdots<i_r\leq n} 2^{r/2} \operatorname{sign}(M_{i_1\cdots i_r})\xi_{i_1}\cdots\xi_{i_r} \\ &\in H^{ev}(\operatorname{Pic}(X); Z) .\end{aligned}$$

Applying the Corollary in § 4.2 we deduce that the numbers $\operatorname{sign}(M_{i_1\cdots i_r})$ are homotopy invariants of X for smooth X. This result has been first proved in some special cases by Novikov [11] and Rohlin [14] and in general by W. C. Hsiang-Farrell [6] and Kasparov [7] using nonsimply connected surgery.

Remarks. (1). The above calculation does not apply, of course, to Poincaré complexes. However, for a Poincaré complex X of formal dimension $2k$, the element $\operatorname{Nov}(X) \in K(\operatorname{Pic}(X))$ is defined (see § 4.3) and we have

$$\operatorname{ch}\operatorname{Nov}(X) = \sum_{\substack{0\leq r\leq n \\ r\equiv 0(2)}} a_{i_1\cdots i_r}\xi_{i_1}\cdots\xi_{i_r} \in H^{ev}(\operatorname{Pic}(X); Z) .$$

($a_{i_1\cdots i_r}$ are integers since for any torus T the Chern character is an isomorphism $\operatorname{ch}\colon K(T) \to H^{ev}(T; Z)$.) One can show that $a_{i_1\cdots i_r} = 0$ unless $r \equiv 2k(4)$. (In fact, consider the involution $j\colon \operatorname{Pic}(X) \to \operatorname{Pic}(X)$ defined by $j(x) = x^{-1}$, $x \in \operatorname{Pic}(X)$. Then the construction in § 4.3 can be done respecting j. It follows that

$$\operatorname{Nov}(X) \in \operatorname{Image}\{KR(\operatorname{Pic}(X), j) \to K(\operatorname{Pic}(X))\} \qquad \text{for } k \text{ even} ,$$

and $\operatorname{Nov}(X)$ is of the form $y - j^! y^*$, $y \in K(\operatorname{Pic}(X))$ for k odd. Our assertion follows then easily.)

It is an interesting problem to decide whether the numbers $a_{i_1 \dots i_r}$ for $r \equiv 2k(4)$ are divisible by $2^{r/2}$ (as in the smooth case) at least in the case when X satisfies Poincaré duality over the integers.

(2). Until now, we have only considered even dimensional manifolds. The odd dimensional case can be reduced to the even dimensional case in the following way: If X is a $(2k - 1)$-manifold, define

$$\mathrm{Nov}\,(X) = \mathrm{Nov}\,(X \times S^1) \in K(\mathrm{Pic}\,(X \times S^1)) = K((\mathrm{Pic}\,(X)) \times S^1)\ .$$

The restriction of this element to $K(\mathrm{Pic}\,(X))$ is zero, so it can actually be considered as an element in $K^1(\mathrm{Pic}\,(X))$. It follows from § 4.2 that $\mathrm{Nov}\,(X)$ is a homotopy invariant of X. A similar remark applies to Poincaré complexes of formal dimension $(2k - 1)$.

4.5. Novikov's higher signature for a family of manifolds. We shall consider a compact connected space Y and a manifold Z over Y with fibre X, a $(2k)$-dimensional manifold (see § 3.1). Define two torus-bundles $\mathrm{Alb}\,(Z/Y)$, $\mathrm{Pic}\,(Z/Y)$ over Y fibrewise. For example, the fibre of $\mathrm{Alb}\,(Z/Y)$ at $y \in Y$ is $\mathrm{Alb}\,(X_y)$ where X_y is the fibre of $Z \to Y$ at $y \in Y$. The structure group of these torus-bundles is the group of automorphisms of $H_1(X; \boldsymbol{Z})/\mathrm{Tors}$; in particular it is a discrete group. The construction in § 4.1 can be done fibrewise and gives a canonical line bundle $\boldsymbol{L}$ over $\mathrm{Alb}\,(Z/Y) \underset{Y}{\times} \mathrm{Pic}\,(Z/Y)$. (The line bundle $\boldsymbol{L}$ is actually induced from a universal situation: The standard action of $GL(n, \boldsymbol{Z})$ on $\boldsymbol{R}^n/\boldsymbol{Z}^n$ gives a torus bundle $\mathscr{T}$ over $B = BGL(n, \boldsymbol{Z})$. If $\mathscr{T}^*$ is the dual torus bundle, there is a universal line bundle $\boldsymbol{L}_u$ over $\mathscr{T} \underset{B}{\times} \mathscr{T}^*$. One can show that

$$H^2(\mathscr{T} \underset{B}{\times} \mathscr{T}^*; \boldsymbol{Q}) \to H^2((\boldsymbol{R}^n/\boldsymbol{Z}^n) \times (\boldsymbol{R}^n/\boldsymbol{Z}^n)^*; \boldsymbol{Q})$$

given by restriction to a fibre is injective with 1-dimensional image. Hence $c_1(\boldsymbol{L}_u)$ is determined by its restriction to a fibre, where it is known by § 4.1. However $c_1(\boldsymbol{L}_u)$ is not decomposable in sum of products of 1-dimensional elements, since $H^1(\mathscr{T} \underset{B}{\times} \mathscr{T}^*; \boldsymbol{Q}) = 0$.)

Suppose for simplicity that X is connected and that a section $Y \to Z$ is given so that each fibre X_y will have a specific base-point. Choose Riemann metrics on X_y depending continuously on $y \in Y$. By the construction in § 4.2 these metrics and the base points determine a fibre-preserving map $\rho: Z \to \mathrm{Alb}\,(Z/Y)$, whose fibre-homotopy type is independent of the choice of metrics. We have then a family of flat hermitian bundles

$$(X \to Z \underset{Y}{\times} \mathrm{Pic}\,(Z/Y) \to \mathrm{Pic}\,(Z/Y),\ (\rho \underset{Y}{\times} 1)^! \boldsymbol{L})\ .$$

The analytic index of the associated family of elliptic operators (see § 3.1) can be called the Novikov higher signature of $Z \to Y$. This is an element $\mathrm{Nov}(Z/Y) \in K(\mathrm{Pic}(Z/Y))$, and again it follows from Theorem 3.3 that it is an invariant under fibre-homotopy equivalences respecting the given section.

4.6. We shall consider now a variant of a situation considered in § 4.5 which will be used in § 4.7 for proving a multiplicativity formula for Novikov's higher signature in fibre bundles. Instead of supposing that $Z \to Y$ has a section we shall suppose that $Z \to Y$ is associated to a principal G-bundle over Y, where G is a simply connected topological group, together with a given homomorphism $h: G \to \mathrm{Diff}_0(X)$. ($\mathrm{Diff}_0(X)$ is the identity component of $\mathrm{Diff}(X)$.) Then there are a finite open covering $\{Y_\alpha\}$ of Y and maps $f_{\alpha\beta}: Y_\alpha \cup Y_\beta \to G$ defined for $Y_\alpha \cap Y_\beta \neq \phi$ such that $f_{\alpha\alpha} = 1$, $f_{\alpha\beta}f_{\beta\gamma}f_{\gamma\alpha} = 1$, and

$$Z = \coprod_\alpha (Y_\alpha \times X)/\sim$$

where $(y, x) \sim (y, h(f_{\alpha\beta}(y))x)$ for $y \in Y_\alpha \cap Y_\beta$, $x \in X$. Let $\tilde{X} \to X$ be a covering of X such that $\boldsymbol{Z}^n$ acts freely on $\tilde{X}$ and $\tilde{X}/\boldsymbol{Z}^n = X$. Let $\mathscr{D}$ be the identity component of the group of diffeomorphisms of $\tilde{X}$ which are $\boldsymbol{Z}^n$-equivariant. Then there is a natural homomorphism $u: \mathscr{D} \to \mathrm{Diff}_0(X)$ which is surjective and has discrete kernel, hence it is a covering. Since G is simply connected there is a unique (continuous) homomorphism $\tilde{h}: G \to \mathscr{D}$ such that $h = u \circ \tilde{h}$. Define

$$\tilde{Z} = \coprod_\alpha (Y_\alpha \times \tilde{X})/\sim$$

where $(y, \tilde{x}) \sim (y, \tilde{h}(f_{\alpha\beta}(y))\tilde{x})$ for $y \in Y_\alpha \cap Y_\beta$, $\tilde{x} \in \tilde{X}$. Then $\boldsymbol{Z}^n$ acts freely on $\boldsymbol{Z}^n$ and $\tilde{Z}/\boldsymbol{Z}^n = Z$. Let T^* be the torus $\mathrm{Hom}(\boldsymbol{Z}^n, S^1)$. For any $t^* \in T^*$ we have a line bundle $L_{t^*} = \tilde{Z} \times_{\boldsymbol{Z}^n} \boldsymbol{C}$ over Z (where $\boldsymbol{Z}^n$ acts on $\boldsymbol{C}$ via t^*). Making t^* vary in T^* we obtain a family of flat hermitian bundles

$$\text{(i)} \qquad (X \to Z \times T^* \to Y \times T^*, L_{(1)}) \,.$$

Similarly, if we start with $X \times Y \to Y$ instead of Z and $\tilde{X} \times Y$ instead of $\tilde{Z}$, we obtain a family of flat hermitian bundles

$$\text{(ii)} \qquad (X \to X \times Y \times T^* \to Y \times T^*, L_{(2)}) \,.$$

Let σ_1, σ_2 be the elements in $K(Y \times T^*)$ associated to (i) and (ii) (see § 3.1). *We shall prove that* $\sigma_1 = \sigma_2$. Let $\tilde{\Omega}_1$, $\tilde{\Omega}_2$ be the complexes of Frechet bundles over $Y \times T^*$ associated to (i) and (ii). By definition (see § 3.1) we have

$$(\tilde{\Omega}_1)_{(y,t^*)} = \tilde{\Omega}(X_y, L_{(1)} \,|\, X_y)) \,, \qquad (\tilde{\Omega}_2)_{(y,t^*)} = \tilde{\Omega}(X, L_{(2)} \,|\, X \times \{y\}) \,.$$

The local trivializations for Z and $\tilde{Z}$ described above define specific isomorphisms

$$\Phi_\alpha : \tilde{\Omega}_1 | Y_\alpha \times T^* \approx \tilde{\Omega}_2 | Y_\alpha \times T^*$$

such that

$$\Phi_\beta \circ \Phi_\alpha^{-1} : \tilde{\Omega}_2 | (Y_\alpha \cap Y_\beta) \times T^* \to \tilde{\Omega}_2 | (Y_\alpha \cap Y_\beta) \times T^*$$

is chain-homotopic to identity for any fixed $(y, t^*) \in (Y_\alpha \cap Y_\beta) \times T^*$ (since G is connected). Let $\{\phi_\alpha\}$ be a partition of unity on Y with $\operatorname{supp} \phi_\alpha \subset Y_\alpha$. Define $\Phi : \tilde{\Omega}_1 \to \tilde{\Omega}_2$ by $\Phi = \sum_\alpha \phi_\alpha \Phi_\alpha$. Then Φ is a morphism of complexes over $Y \times T^*$, which is a chain-homotopy equivalence for any fixed $(y, t^*) \in Y \times T^*$. (This follows from the fact that Φ_α is chain-homotopic to Φ_β over (y, t^*), $y \in Y_\alpha \cap Y_\beta$, $t^* \in T^*$.) Also it is clear that for any fixed (y, t^*), the map induced by Φ in cohomology is compatible with the cup-products. Let $\mathscr{C}_1 \underset{p_1}{\overset{q_1}{\rightleftarrows}} \tilde{\Omega}_1$, $\mathscr{C}_2 \underset{p_2}{\overset{q_2}{\rightleftarrows}} \tilde{\Omega}_2$ be as in Proposition 3.2. Then $p_2 \Phi q_1 : \mathscr{C}_1 \to \mathscr{C}_2$ is a homotopy equivalence of regular hermitian complexes over $Y \times T^*$ (see § 1.2). By Proposition 1.6 $\sigma(\mathscr{C}_1) = \sigma(\mathscr{C}_2)$. But by definition (see § 3.2) we have $\sigma_1 = \sigma(\mathscr{C}_1)$, $\sigma_2 = \sigma(\mathscr{C}_2)$. It follows that $\sigma_1 = \sigma_2$ which proves our assertion. (Note the similarity of this proof with the one of Theorem 3.3.)

The interest of this equality lies in the fact that σ_2 is clearly in the image of the homomorphism $K(T^*) \to K(Y \times T^*)$ induced by the projection $Y \times T^* \to T^*$. Hence the same is true for σ_1.

We shall express this fact in terms of $\operatorname{ch} \sigma_1$. The covering $\tilde{X} \to X$ is induced by a map $X \xrightarrow{\rho_1} T = \boldsymbol{R}^n / \boldsymbol{Z}^n$. Let $z_1, \cdots, z_m$ be a basis of $H^1(T; \boldsymbol{Z})$, and $\zeta_1, \cdots, \zeta_m$ the dual basis of $H^1(T^*; \boldsymbol{Z})$. Then

$$x_i = \rho_1^*(z_i) \in H^1(X; \boldsymbol{Z}) = H^1(Z; \boldsymbol{Z}) .$$

We have the following formula (compare § 3.3 and § 4.4):

$$\operatorname{ch} \sigma_1 = (-1)^k \sum_{0 \le r \le m} \sum_{0 \le i_1 < \cdots < i_r \le m} \pi_*(\tilde{\mathscr{L}}(Z/Y) x_{i_1} \cdots x_{i_r}) \zeta_{i_1} \cdots \zeta_{i_r}$$
$$\in H^{ev}(Y \times T^*; \boldsymbol{Q}) .$$

Here $\pi : Z \to Y$ is the projection, and π_* denotes integration along fibres. Since any set $\{x_1, \cdots, x_n\}$ of elements in $H^1(X; \boldsymbol{Z})$ is induced from the generators of H^1 of some torus by a suitable map $X \to T$ we get

Theorem. *Suppose that the structure group of $X \to Z \to Y$ is simply connected, and let $x_1, \cdots, x_n \in H^1(X; \boldsymbol{Z}) = H^1(Z; \boldsymbol{Z})$. Then $\pi_*(\tilde{\mathscr{L}}(Z/Y) x_1 \cdots x_n) \in H^*(Y; \boldsymbol{Q})$ is zero in all positive degrees; in other words it is equal to*

$$\tilde{\mathscr{L}}(X) x_1 \cdots x_n [X] \cdot 1 \in H^0(Y; \boldsymbol{Q}) .$$

4.7. Multiplicativity in fibre-bundles.

Theorem. *Suppose that $X \to Z \to Y$ is a smooth fibre bundle with X, Y, Z manifolds of even dimension. If Y is 2-connected, then one can identify* Pic (X) *and* Pic (Z), *and the formula*

$$(*) \qquad \operatorname{Nov}(Z) = \operatorname{Nov}(X) \cdot \operatorname{sign}(Y)$$

holds in $K(\operatorname{Pic}(X))$.

Proof. Let $x_1, \cdots, x_n$ be a basis of $H^1(X; Q) = H^1(Z; Q)$, and $\xi_1, \cdots, \xi_n$ the dual basis of $H^1(\operatorname{Pic}(X); Q) = H^1(\operatorname{Pic}(Z); Q)$. We have

$$\begin{aligned}
&\operatorname{ch} \operatorname{Nov}(Z) \\
&= (-1)^{(\dim Z)/2} \sum_{0 \leq r \leq n} \sum_{0 \leq i_1 < \cdots < i_r \leq n} \tilde{\mathscr{L}}(Z) x_{i_1} \cdots x_{i_r}[Z] \xi_{i_1} \cdots \xi_{i_r} \qquad \text{(by 4.4)} \\
&= (-1)^{(\dim Z)/2} \sum \sum \pi^*(\tilde{\mathscr{L}}(Y) \cdot \tilde{\mathscr{L}}(Z/Y) x_{i_1} \cdots x_{i_r}[Z] \xi_{i_1} \cdots \xi_{i_r} \\
&\qquad \text{(we omit summation indices)} \\
&= (-1)^{(\dim Z)/2} \sum \sum \pi_*(\tilde{\mathscr{L}}(Z/Y) x_{i_1} \cdots x_{i_r}) \tilde{\mathscr{L}}(Y)[Y] \xi_{i_1} \cdots \xi_{i_r} \\
&= (-1)^{(\dim Z)/2} \sum \sum \tilde{\mathscr{L}}(X) x_{i_1} \cdots x_{i_r}[X] \tilde{\mathscr{L}}(Y)[Y] \xi_{i_1} \cdots \xi_{i_r} \qquad \text{(by 4.6)} \\
&= (-1)^{(\dim Y)/2} \operatorname{ch} \operatorname{Nov}(X) \cdot \operatorname{sign}(Y) .
\end{aligned}$$

Since for $\dim Y = \text{odd}$, both sides in (*) are zero, the Theorem follows.

This proof is modeled on Atiyah's proof of multiplicativity of the Hirzebruch signature [1, § 4] in case the base is 1-connected.

5. Non-abelian fundamental groups

In this section we will show how the results of § 2 combined with a result of Matsushima lead to homotopy invariance properties of the Pontrjagin classes of certain $K(\pi, 1)$-manifolds, with highly non-abelian fundamental group π.

5.1. Let $G = \operatorname{Sp}(2n, \boldsymbol{R})$ be the real symplectic group in $2n$ variables, i.e., the group of automorphisms of $\boldsymbol{R}^{2n}$ preserving the alternate form $\sum_{i=1}^{n} (x_i y_{n+i} - x_{n+i} y_i)$.

Denote by $\mathscr{R}$ the Grothendieck group generated by all isomorphism classes of finite dimensional complex (continuous) representations of G with a fixed nondegenerate hermitian form, invariant under the action of G. For any such representations E, F we have the relation $[E \oplus F] = [E] + [F]$ in $\mathscr{R}$, where $[E]$ is the class of E in $\mathscr{R}$, and $E \oplus F$ is the orthogonal direct sum of E and F. $\mathscr{R}$ can be made into a λ-ring in the usual way.

G acts naturally in $\boldsymbol{C}^{2n}$, considered with the hermitian form $\sum_{i=1}^{n} (x_i \overline{y_{n+i}} - x_{n+i} \bar{y}_i)$, of type (n, n). We shall denote this particular representation of G by E_0. Let E be a representation space of G with a nondegenerate invariant hermitian form.

Restricting this representation to the maximal compact subgroup $U(n) \subset G$, E splits into orthogonal direct sum $E = E^+ \oplus E^-$ such that E^+, E^- are invariant under $U(n)$ and the herimitian form is positive (resp. negative) definite on E^+ (resp. E^-). Hence the associated hermitian bundle $\phi(E)$ over the classifying space BG has a corresponding splitting

$$\phi(E) = \phi(E)^+ \oplus \phi(E)^- .$$

(Note that BG is homotopy equivalent to $BU(n)$.)

By taking Chern character we get an element $\mathrm{ch}\,(\phi(E)^+ - \phi(E)^-) \in H^*(BG)$. Here, and in the remainder of this section, cohomology is taken with rational coefficients and $H^* = \prod_k H^k$.

The correspondence $E \to \mathrm{ch}\,(\phi(E)^+ - \phi(E)^-)$ defines a ring homomorphism $\psi\colon \mathscr{R} \to H^*(BG)$, respecting 1. Consider, in particular, the representation E_0. Let $x_1, \cdots, x_n$ denote the roots of $\phi(E_0)^+$. Then it is easy to see that

$$\psi(E_0) = \sum_{i=1}^{n} (e^{x_i} - e^{-x_i}) \in H^*(BG) .$$

It is well known that $H^*(BG)$ can be identified with the algebra of symmetric formal series over Q, in the variables $x_1, \cdots, x_n$, and as such it has a natural topology.

Let σ_i ($i = 1, \cdots, n$) be the i-th elementary symmetric function in the variables $e^{x_1} - e^{-x_1}, \cdots, e^{x_n} - e^{-x_n}$. In particular $\psi(E_0) = \sigma_1$. One checks easily that for any i, $1 \leq i \leq n$, one has

$$\psi(\Lambda^i E_0) = \sigma_i + a_{i,2}\sigma_{i-2} + a_{i,4}\sigma_{i-4} + \cdots ,$$

where $a_{i,2}, a_{i,4}, \cdots$ are integers. It follows by induction that $\sigma_1, \cdots, \sigma_n$ belong to the image of ψ. Since ψ is a ring homomorphism, the image of ψ will contain any symmetric polynomial in $e^{x_1} - e^{-x_1}, \cdots, e^{x_n} - e^{-x_n}$. Note that $e^x - e^{-x} = 2x +$ higher terms, hence $x = \sum_{i=1}^{\infty} a_i(e^x - e^{-x})^i$, $a_i \in Q$. From this it follows that the symmetric polynomials with rational coefficients in $e^{x_1} - e^{-x_1}, \cdots, e^{x_n} - e^{-x_n}$ are dense in $H^*(BG)$. Hence we have proved the

Proposition. *The image of the ring homomorphism* $\psi \otimes 1_Q\colon \mathscr{R} \otimes Q \to H^*(BG)$ *is dense in* $H^*(BG)$.

5.2. Proposition. *Let X, X' be two connected $2k$-manifolds, and $f\colon X' \to X$ an orientation preserving smooth homotopy equivalence. Fix a base point $x_0 \in X$, and consider a homomorphism $\rho\colon \pi_1(X, x_0) \to G$. Let h denote the composition $X \to B\pi_1(X, x_0) \to BG$ defined up to homotopy. Then for any class $u \in H^*(BG)$ one has*

$$\tilde{\mathscr{L}}(X)h^*(u)[X] = \tilde{\mathscr{L}}(X')f^*h^*(u)[X'] .$$

Proof. By § 5.1, for any $u \in H^*(BG)$ there exist an integer $n > 0$ and two representations E and F of G with invariant nondegenerate hermitian forms such that

$$nu - \mathrm{ch}\,(\phi(E)^+ - \phi(E)^-) + \mathrm{ch}\,(\phi(F)^+ - \phi(F)^-)$$

is zero in degrees $\leq \dim X$. In particular,

$$(*) \quad h^*(nu) = h^*\,\mathrm{ch}\,(\phi(E)^+ - \phi(E)^-) - h^*\,\mathrm{ch}\,(\phi(F)^+ - \phi(F)^-) \in H^*(X)\ .$$

E and F give rise, via ρ to flat hermitian bundles on X. Applying § 2.2 to these flat bundles and using (*) we get the proposition.

5.3. The following result has been brought to the author's attention by A. Borel.

Theorem (Matsushima [9]). *Let Γ be a discrete, torsion free subgroup of G such that G/Γ is compact. Then the homomorphism $H^p(BG) \to H^p(B\Gamma)$ induced by the inclusion $\Gamma \subset G$ is surjective for $p < (n+2)/4$.*

Remark. A. Borel has recently proved an extension of this result in the case where Γ is any arithmetic subgroup in G.

5.4. Let Γ be as in § 5.3. Then the double coset space $X = \Gamma \backslash G/U(n)$, where $U(n)$ is a maximal compact subgroup of G, is a manifold of dimension $n(n+1)$, which is a $K(\Gamma, 1)$.

Theorem. *Let X' be a manifold, and $f\colon X' \to X$ an orientation preserving homotopy equivalence (X is the $K(\Gamma, 1)$ described above). Then $\tilde{\mathscr{L}}_{4k}(X') = f^*\tilde{\mathscr{L}}_{4k}(X)$ for $4k > n(n+1) - (n+2)/4$.*

Proof. In fact, for any class $v \in H^{n(n+1)-4k}(X)$, $4k$ as above, we have

$$\tilde{\mathscr{L}}_{4k}(X')f^*(v)[X'] = \tilde{\mathscr{L}}_{4k}(X)v[X]$$

using Proposition 5.2 and § 5.3. Since $[X] = f_*[X']$, the theorem follows from Poincaré duality.

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